The Nonconforming Point Interpolation Method applied to Electromagnetic Problems

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Abstract — Nonconforming Point Interpolation Method (NPIM) is a meshless method that has been applied to problems in Mechanics in the last years. In this paper, we investigate NPIM in Electromagnetism. We present its formulation and shape functions, which are generated by the radial point interpolation method with polynomial terms. The numerical results are compared to the ones obtained by the Finite Element Method (FEM).

I. INTRODUCTION

Meshless (or meshfree) methods are an alternative to traditional numerical techniques such as the Finite Element Method (FEM) and the Finite Difference Method (FDM). In general, meshfree methods do not use a mesh for shape function generation neither for integration of the weak form [1]. If a mesh is yet necessary then it is called a background mesh (or background grid) and it is used only for integration, leading to weaker requirements on its quality.

There are several meshless methods developed so far [1] such as the Smoothed particle hydrodynamics (SPH), the Element-free Galerkin (EFG), the Meshless Local-Petrov Galerkin (MLPG), the Point Interpolation Method (PIM) family, and others. Under the PIM family we can cite the Local Point Interpolation Method (LPIM) [1], the Conforming Point Interpolation Method (CPIM) [1], and the Nonconforming Point Interpolation Method (NPIM) [1]. Most of those methods have already been applied to electromagnetic problems. EFG might be the most used method since it was one of the first meshless methods to arise. In [2] EFG is applied to a three-dimensional electrostatic problem. In [3] MLPG is used to solve a microwave guide problem. LPIM in [4] solves a problem that involves Eddy-Current.

As NPIM has not yet been tested with electromagnetic problems, it is important to evaluate its performance on electromagnetic field computation. For so, in this paper we use NPIM to solve two-dimensional static problems. We introduce the PIM shape function generation, the problem mathematical formulation and the weak form integration for NPIM in the following section. Numerical results are then presented and compared against FEM. It is shown that the method converges and that the results have the same numerical quality as the ones generated by FEM.

II. MATHEMATICAL FORMULATION

A. Point Interpolation Method

The Point Interpolation Method was proposed to replace Moving Least Squares (MLS) approximation for creating shape functions. The major advantages of PIM are the excellent accuracy in function fitting and that the created shape functions possess the Kronecker delta function property, which allows simple imposition of essential boundary conditions as in the conventional FEM [1].

The approximation properties lie on the basis functions used for the shape functions construction. The present work uses Radial Point Interpolation with polynomials (RPIMp) in basis, which means that both radial basis functions (RBF) and polynomial terms are used.

The approximation $u^h(\boldsymbol{x})$ at a point $\boldsymbol{x}$ is given by

$$u^h(\boldsymbol{x}) = \sum_{i=1}^{n} R_i(\boldsymbol{x})a_i + \sum_{j=1}^{m} p_j(\boldsymbol{x})b_j \quad (1)$$

where n is the number of nodes in the support domain of $\boldsymbol{x}$, m is the number of polynomial terms used in basis, a_i's are the coefficients for radial basis $R_i(\boldsymbol{x})$ and b_j's the coefficients for polynomials terms $p_j(\boldsymbol{x})$. The coefficients a_i and b_j are determined enforcing interpolation passing through all the n nodes within the support domain [1].

An interesting characteristic of RPIMp is the assurance that the shape functions will always exist and that consistency is achieved according to the polynomial basis [1]. On the other hand, RPIMp functions are not compatible because no weight function (as in MLS) is used in their construction. The approximation thus can be discontinuous when the support domain changes while the point of interest moves. The nodes in the support domain are updated suddenly, meaning that when the nodes are entering or leaving the support domain, they are actually "jumping" into or out of the support domain. Therefore, the function approximated using the RPIMp shape functions can jump [1]. In MLS, this "jumping" is avoided by the presence of the weight function that makes the nodes enter or leave the moving support domain in a smooth manner.

Because the PIM shape functions are not compatible, depending on the energy principles used in the method formulation, PIM can be conforming or nonconforming [1]. When constraints are used to enforce compatibility, the method will be conforming. NPIM ignores the compatibility issues and impose no constraints in the mathematical formulation which makes it nonconforming. Nevertheless, NPIM converges to the exact solution even with compatibility issues [1].

B. Weak Form

Electrostatic problems can be described by the following boundary value problem:

$$\nabla \cdot (\varepsilon \nabla V) = -\rho \quad (2)$$

$$V = g \text{ on } \Gamma_g \quad \text{and} \quad -\varepsilon\frac{\partial V}{\partial n} = h \text{ on } \Gamma_h \quad (3)$$

where V is the electrostatic potential, ε is the electric permittivity, ρ is the space charge density, g is the electric potential imposed on the Dirichlet boundary Γ_g and h is the value imposed on the Neumann boundary Γ_h.

Using the weighted residual method with an arbitrary test function w, we obtain from (2) and (3) the following weak form

$$\int_\Omega \varepsilon\nabla w \cdot \nabla V d\Omega = \int_\Omega w\rho \, d\Omega - \int_{\Gamma_h} wh d\Gamma \,. \quad (4)$$

Applying the Galerkin method on (4), we get the equation system in matrix form

$$KV = F \quad (5)$$

where

$$K_{ij} = \int \varepsilon\nabla\phi_i \cdot \nabla\phi_j d\Omega \quad \text{and} \quad F_i = \int_\Omega \phi_i\rho \, d\Omega - \int_{\Gamma_h} \phi_i h d\Gamma \quad (6)$$

with ϕ_i as the PIM shape functions created by manipulation of (1) as described in [1]. It should be noted that no method to impose essential boundary conditions is used once PIM shape functions have the Kronecker delta property. Also, there are no constraints to enforce compatibility in (6). We can see that EFG and NPIM are quite similar and they basically diverge in the shape function generation. EFG creates shape functions using MLS, and they are compatible but do not have Kronecker delta property, so essential boundary conditions must be enforced in the weak form (4). On the other hand, in NPIM essential boundary conditions are naturally imposed but its shape functions are not compatible.

C. Integration

The integration in NPIM is done with a background grid (or mesh) over the domain in the same way as in EFG. The cell grid can be of any shape but rectangular or triangular are usually adopted due to their simplicity. The integration process is often carried out with Gauss quadrature over each cell. In this paper, rectangular cell grid is used.

III. NUMERICAL RESULTS AND CONCLUSIONS

The accuracy of the method was investigated on an electrostatic problem shown in Fig. 1.

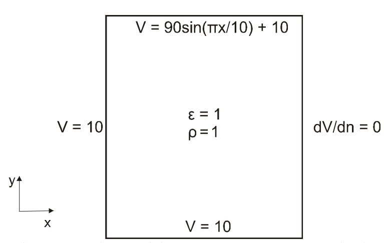

Fig. 1. Electrostatic problem geometry. Domain is 5 x 10.

To solve the problem, it is used 21 x 41 equally spaced nodes. The background grid has 20 x 40 cells. Integration is performed with 16 and 5 quadrature points per cell for domain and boundary integrations, respectively. Linear polynomials and cubic spline RBF [1] are employed in RPIMp. The analytical solution is

$$V_a = \frac{90\sin(\pi x/10)\sinh(\pi y/10)}{\sinh(\pi)} + 10 \,. \quad (7)$$

The meshless method framework presented in [5] is used to implement the method and Fig. 2 shows the solution for NPIM. We can see that the method approximates well the analytical solution, with a relative error order of 10^{-5}. The convergence rate is also found and compared against FEM (Fig. 3). Both methods have the same convergence rate (about 2.0), although NPIM is nonconforming. On the other hand, NPIM presented better accuracy for all used nodes distributions, which started from 66 up to 51681 equally spaced nodes.

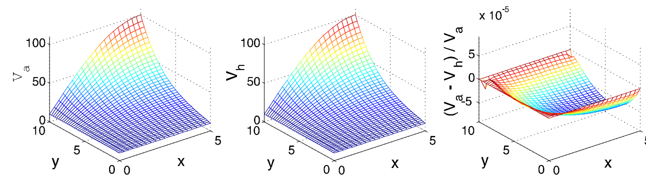

Fig. 2. (a) NPIM solution V_h, (b) analytical solution V_a, and (c) relative error surface.

We conclude that NPIM is a good alternative to solve electromagnetic problems and it has a simple implementation given its formulation and shape function Kronecker delta property characteristic. NPIM shows similar accuracy to FEM and converges to the exact solution.

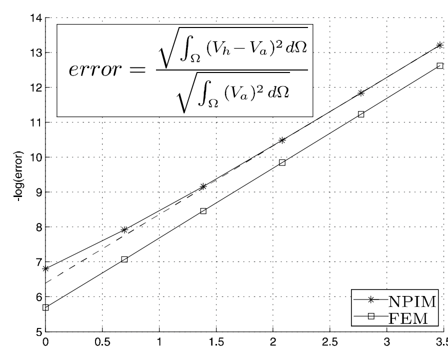

Fig. 3. Error norm for NPIM and FEM. h is the distance among nodes. The convergence rate is calculated using the last points.

IV. REFERENCES

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